Stochastics of radioactive emission

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Statistics of exponentially decaying radiation of any nuclide always obeys the binomial stochastic distribution. The variance is derived for the number of atoms disintegrating in a time interval. The variance determines the error of the disintegrations number. When the measuring time is short compared with the half life then the Poisson stochastics is valid. Today computers can be used to determine either distribution very easily. The dependence of absolute and relative error on the measuring time interval is examined. When you have a short-lived nuclide, stop the counting in time.

Keywords: radioactivity; disintegration statistics; background; error

1 Introduction

Assume that N is the number of atoms of a certain radioactive nuclide at time $t = 0$. Today N may be observed sometimes even exactly. We start with the known experimental law:

$$(dN/N)/dt = -\lambda, \tag{1}$$

for the relative diminishing of the number N, where λ is the disintegration constant. λ is influenced by the chemical effects: in the EC and IC disintegrations, etc., [1, Ch. 12]. During a time interval T the number of the disintegrated nuclides is

$$n = N(1 - \exp(-\lambda T)). \tag{2}$$

The stochastics of n, its statistics and the influence of the measuring time T, the author treats thoroughly in [2]. Also here we suppose that all the nuclide

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disintegrations n are detected. We suppose that the count dead time and the background are zero (except in Ch.3). We have

$$\lim_{T\to\infty} n = N.$$

At this limit $T \to \infty$, then $N - n$ is smaller than 1 [2, Fig.1]. There n is very deterministic, i.e., its statistical error is very small.

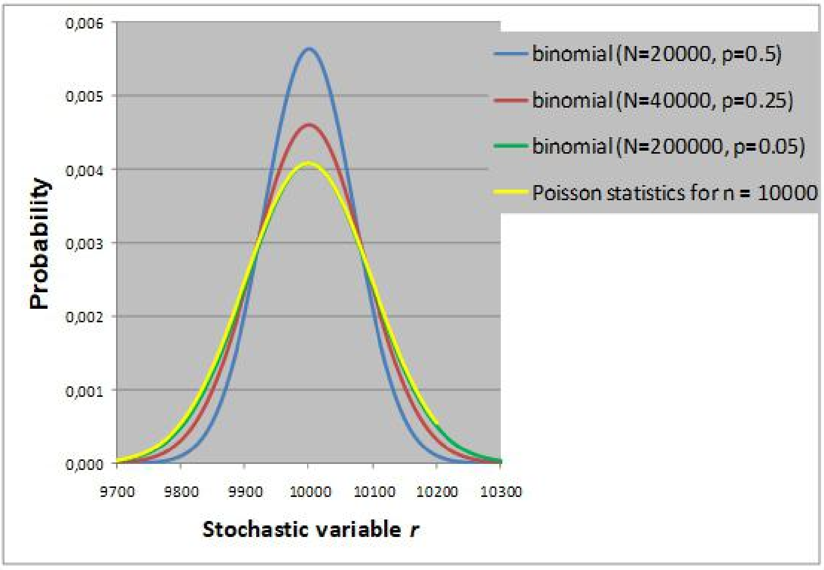

Figure 1: Binomian and Poisson statistics. r is the random number of disintegrations in time T. Distributions for the mean value $n = Np = 10000$. $p = 1\exp(-\lambda T)$

It has been shown that the probability of radioactive disintegration is always described by the binomial distribution law [1,2,3,4,5]. The introduction is presented in [6,7]. Its Poisson approximation demands $\lambda T \ll 1$. Fig.1 illustrates the distributions. When $p = 0.5$ then $T = t_{1/2}$. The modern computers calculate today the binomial and Poisson distributions easily, even when N and n are large. In Fig. 1 we have $n = pN = 10000$, and we present the binomial distributions

$$P(r) = \binom{N}{r} p^r (1-p)^{N-r} = \binom{N}{r} (1 - e^{-\lambda T})^r (e^{-\lambda T})^{N-r}$$

as well as (when $N \to \infty$ and $p \to 0$ or $\lambda T \to 0$ with $pN = n$) the Poisson distribution

$$P(r) = \frac{n^r}{n!} e^{-n}.$$

Here r is the random variable of the number of disintegrations. For ^{14}C as well as for ^{40}K I have calculated equivalent stochastic distributions binomially (N estimated) and when using Poisson presentation.

In [2] have been postulated: the probability $p = 1 - \exp(-\lambda T)$ for a radioactive nuclide to disintegrate during the time T. For the mean value of r we find the estimate

$$E(r) = N(1 - \exp(-\lambda T) = n. \tag{3}$$

i.e., the value of (2), so that the postulation seems good. The variance of r is[2]

$$Var(r) = \sigma^2 = Npq = n \exp(-\lambda T). \tag{4}$$

The paper [2] has the detailed derivations, elucidating all the premises for this stochastic model.

2 Error considerations

We have, see (4), the standard deviation for n:

$$\epsilon = \Delta n = \sigma(r) = \sqrt{n} \exp(-\lambda T/2) = (N(1 - \exp(-\lambda T)))^{\frac{1}{2}} \exp(-\lambda T/2). \tag{5}$$

We present the error $\epsilon(T) = \Delta n$ in Fig. 2 for $n = 10000$. At $T = t_{1/2}$ $\epsilon(T)$

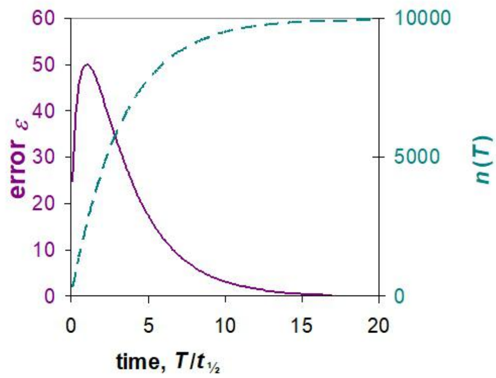

Figure 2: Error of n (the number of disintegrations) when $N = 10000$, and the curve $n(T)$, Eq. (2).

has the maximum $\sqrt{N}/2$ (50 in Fig. 2), and for any N $\epsilon/N^{\frac{1}{2}} = 1/2$ when

[2] In [2] the formula (7) should have $Var(\nu) = E(\nu - \overline{\nu})^2 = pq = (1 - \exp(-\lambda T))exp(-\lambda T)$.

$T/t_{1/2} = 1$. You see that ϵ goes to the value 0 (or the background value) when $T \to \infty$. We seek the time T when (5) has the value ϵ. Surely one such a time is short. The other solution of (5) is

$$T = t_{1/2}(\ln((N/2\epsilon^2)(1 + \sqrt{1 - 4\epsilon^2}))/\ln 2. \tag{6}$$

When ϵ is small then the time T is that where $n \equiv N$. In the Table 1 we have T when $\epsilon = \Delta n = \pm 1$ for some N, and the time when $n(t) = N - 1$.

Table 1. Measuring times T for $\Delta n = \pm 1$ and time t of $n(t) = N - 1$		
N	$T/t_{1/2}$ for $\Delta n = \pm 1$	$t/t_{1/2}$ for $n = N - 1$
100	6.629	6.644
10^4	13.2876	13.2877
10^6	19,93157	19.93157

$$\Delta n/n = \exp(-\lambda T/2)/\sqrt{n} = N^{-1/2}(\exp(\lambda T) - 1)^{-1/2}$$

is important in practice. $\Delta n(T)$ has the maximum, Fig. 2, but the relative error $\Delta n/n$ decreases with T, see Fig. 3, but soon the background is significant, see section 4. Error of n (the number of disintegrations) when N = 10000, and the curve n(T), Eq. (2).

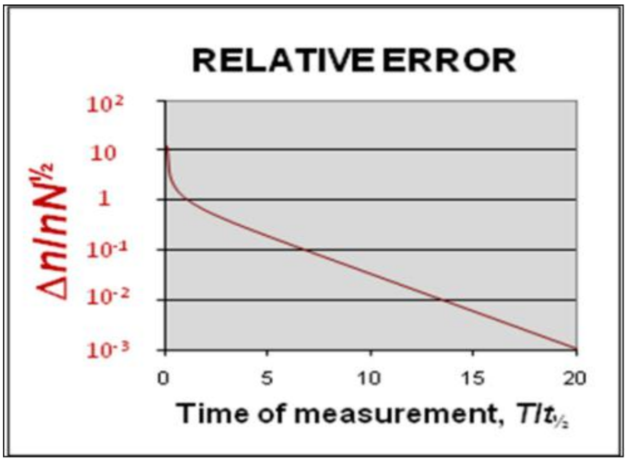

Figure 3: $\Delta n/n$ when multiplied by $\sqrt{N}$.

3 Bayesian approaches

Generally N is unknown. You have a measurement with the result n. What is the result of many equal measurements? Rainwater and Wu [4] consider the Poisson case $\lambda T \ll 1$ and found $E(r) = n + 1$, and $Var(r) = n + 1$. Stevenson [5] has derived the solutions for the binomial distribution. He set $N \geq n$ and each N has the same probability. His result is

$$E(r) = n + 1 - p. \quad (7)$$

In the Poisson case $p \equiv 0$. When $p = 1$ the result agrees with (3). When approaching this limit then T is much larger than $t_{1/2}$. He found $Var(r) = (1 - p)(n + 1)$. $p \to 1(T \to \infty)$ mea¡ns that n approanches the exact value N.In the book [3] of the group of Russian academician Gol'danskii they have similar results. Generally we have $E(r) \in [n, n + 1]$.

4 Background consideration

When the background includes only the counts of long-lived radioactivity, then the background always has the Poisson statistics [2]. In Fig.4 is the

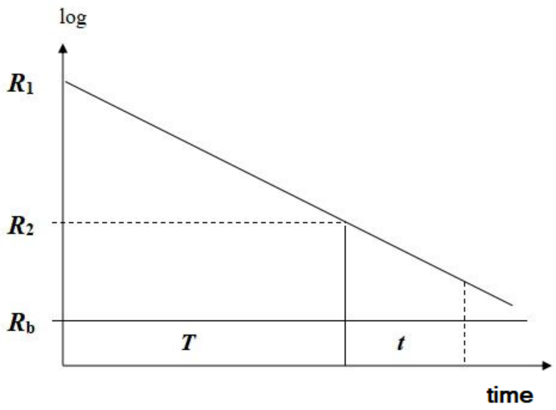

Figure 4: A schema for seeking optimal counting time.

case, where R_1 and R_2 are counting rates, I assume, measured values.There the background rate R_b is supposed to be at permanent level. After counting the time T you maybe want to continue the counting, but how long? There is $R_2 = R_1 \exp(-\lambda T) = \lambda n/(\exp(\lambda T) - 1)$, so long as R_b is not significant. n is the number of counts when counting the time T. The counting rate $R(t) = R_2 \exp(-\lambda t)$. When counting a condition for the time t is obtained by $R(t) = 2R_b$. Longer the t should be surely not

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