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ANALYSIS, CALCULATIONS AND MEASUREMENTS CONCERNING THE MOISTURE MEASURING BY THE NEUTRON METHOD * **

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This paper describes firstly a method of numerical analysis for the moisture measurement with neutrons, with special consideration of the effects of the dry density and the possible boron content of the material. Then a series of measurements performed by using an instrument including an additional iron reflector are rapported and the results analyzed. Finally some theoretical approaches of a more analytical nature are illustrated by applying simple physical models. Special attention is paid to the influence of a nonuniform distribution of moisture on the mathematical treatment.

Sections 2 and 3 are contributions of the first author, section 4 of the second author.

1. INTRODUCTION

The measurement of moisture with neutrons is based on the great neutron moderating power of hydrogen. The effect of the slowing-down upon the neutron densities in the vicinity of a point source of fast neutrons is twofold: the density of slow neutrons increases and the density of fast neutrons decreases. Therefore, by making observations on the spatial distributions of either slow or fast neutrons one can draw conclusions concerning the moisture in the medium. The measuring device consists of a point source of fast neutrons and a detector of slow or fast neutrons. A fast neutron detector is essentially a slow neutron detector surrounded by a block of paraffin or other hydrogenous material that converts the fast neutrons into thermal neutrons for the detection. So far, the neutron method has had its widest use in the measurement of the moisture of soil. Here two alternative arrangements have been employed. In the so-called depth measurement a probe including the source and a thermal detector is lowered into a hole in the soil, whereas in the surface measurement the instrument containing the source and the detector is placed in the vicinity of the surface of the medium to be investigated. The numerous publications rapporting on these measurements are collected in the reference lists of the articles [1] and [2]. In the moisture measurement of concrete slabs, walls, etc. the surface method is most easily applicable (see Pawlin and Spinks [3]), and two different alternatives can here be used. The surface measurement is called a reflection or a transmission experiment depending on whether the source and the detector are on the same side or on the opposite sides of the slab, respectively. The transmission method cannot always be used, but its advantage as compared with the reflection measurement lies generally in its insensitivity to nonuniformities in the spatial distribution of moisture. The reflected fast intensity is usually extremely small so that the fast reflection experiment can hardly be used. The fast transmission measurement, if it is in practice possible (and if the slab is not too thick), is instead very comfortable because it essentially constitutes a monoenergetic problem and thus quite elementary description of the neutron transport is sufficient to give a good agreement between theory and experiment. The measurements of the reflection and the transmission of the slow neutrons produced inside the hydrogenous slab are theoretically equivalent and their analysis is complicated because the whole slowing-down process must be described. On the other hand, the slow reflection principle is the easiest one for practical performance.

It is worth mentioning that as well the crystallized water as the loose water take part in the slowing-down of neutrons in concrete. Conse-

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quently, in comparison of the measurements on different walls one gets information concerning the total amount of water.

2. NUMERICAL ANALYSIS AND THE EFFECTS OF DRY DENSITY AND BORON

2.1. *The numerical method of computation*

The neutron sources most frequently used for these purposes are beryllium sources, and most of them the Ra-Be source despite its high gamma background. The Am-Be and Ac-Be sources are more advantageous in this respect, and their prices are of the same order as those of the Ra-Be source. In our calculations we have used the spectrum of the Ac-Be source [4] illustrated in fig. 2 including a peak at the energy of 0.1 MeV which contains 10% of all source neutrons, but the spectra of the other sources mentioned above are not very different.

The simple age-diffusion model proves to be unsatisfactory in the quantitative analysis of the neutron moisture measuring system when the moisture content exceeds 15% by volume, as is shown in the work by Westmeyer [5]. However, this method may be used, in the same way as the ray-theory described in section 4, for the qualitative investigation of some important effects, such as the influence of the composition and the density of the material. This has been done for instance by Semmler [2]. We have performed numerical calculations of the spatial neutron distributions for a model geometry consisting of a point source in an infinite medium. This situation is close to that of the depth measurement, but the tendencies shown by the results are probably common to all geometries. The code prepared for the computations [14] is based on a combination of the Monte Carlo technique with the method of Selengut and Goertzel. By using the Monte Carlo method [6] in this context it is possible to take into account the elastic and the very important inelastic scattering of neutrons by the atoms of heavy elements present in the material. The neutron events are followed by Monte Carlo until the neutron energy is below 0.75 MeV. The neutrons that have passed this energy value are divided further into two groups separated by the energy value 0.1 MeV. The lower group still contains the neutrons from the 0.1 MeV peak in the Ac-Be spectrum which have had three scattering events. Below the energy 0.75 MeV the diffusion of neutrons is described by a ninegroup diffusion approximation in the Selengut-Goertzel modification ([7], pp. 125-135). The width of the seven highest groups used is two lethargy units, the eighth group above 0.1 eV is a little shorter, and the thermal group consists of the neutrons below 0.1 eV. The two Monte Carlo groups serve as source terms for the two highest diffusion groups. The results of our computations are the spatial distributions of the epithermal flux ($E \approx 1.6$ eV) and the thermal flux. If we omit the errors near the origo introduced by certain oscillations in the calculations at very low moisture, a good agreement with experimental results is to be expected.

2.2. *The results of the computations*

The composition of our test material was: O 48%, MgAl 9%, Si 32%, KCa 5.4% and Fe 3.3% by weight. In the analysis the elements Mg and Al, and, respectively, K and Ca have been identified on the basis of the similarity of their neutron physical properties. The test values used for the density of this medium are 0.6 g/cm^3 and 2.0 g/cm^3, and those for the water content of 0%, 5%, 20% and 45% by volume.

As a result of the Monte Carlo calculation we present the function $4\pi r^2 S$, where S is the density of neutrons with an energy of 0.3 MeV, as a function of r, at different densities and water contents (fig. 1). The effect of the density upon

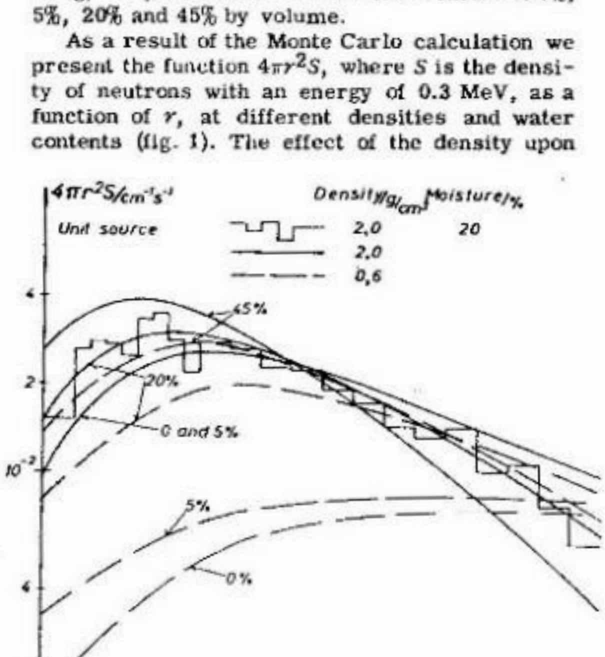

Fig. 1. Neutron distributions at energy 0.3 MeV.

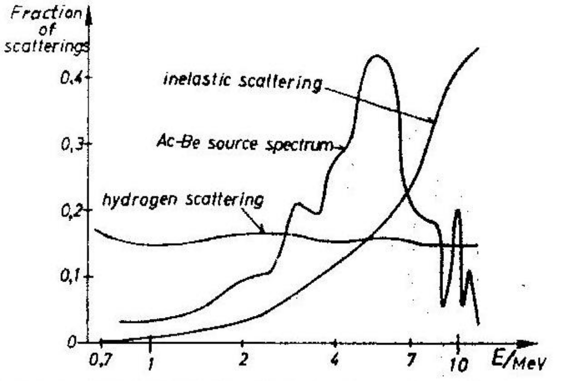

Fig. 2. Comparison of source spectrum at scatterings when density is 2.0 g/cm^3 and moisture 20%.

these results is remarkable. The difference between the density functions for dry material at 0.6 g/cm^3 and 2.0 g/cm^3 is a little greater than the difference between the curves at the water contents of 0% and 20% when the density is 0.6 g/cm^3. The reason for this effect is illustrated in fig. 2. Here the source spectrum and the relative probabilities for elastic scattering in hydrogen and inelastic scattering in the material are presented as functions of the energy. At the average source energy the probabilities for hydrogen and inelastic scattering are nearly equal, but even then the inelast.c scattering is dominating in the slowing-down of neutrons. This is due to the fact that the average energy loss per inelastic scattering is about 90% of the initial energy, as compared the corresponding number, 50% for the proton scattering. The hydrogen scattering dominates only below ca. 1 MeV. Hence we can conclude that the neutron energies of the usual sources are too high for an effective measurement. An ideal source is found not to exist, the most suitable spectrum is that of the Ra-γ-Be photoneutron source, which however has a very high gamma background. Other sources of possible use are the Po-Li source and the spontaneous fission sources.

It is apparent from the results of the multigroup calculations that, at a fixed moisture, the difference between the flux distributions corresponding to different dry densities is at the epithermal energy greater than at thermal energies. Therefore it is impossible to avoid the effect of the density by using epithermal detectors. The thermal flux Φ as a function of distance from the source is plotted in fig. 3. Fig. 4 shows the dependence of the thermal flux in the origo on the moisture at different dry densities of the material. In practice such curves that serve as calibration curves of the instrument

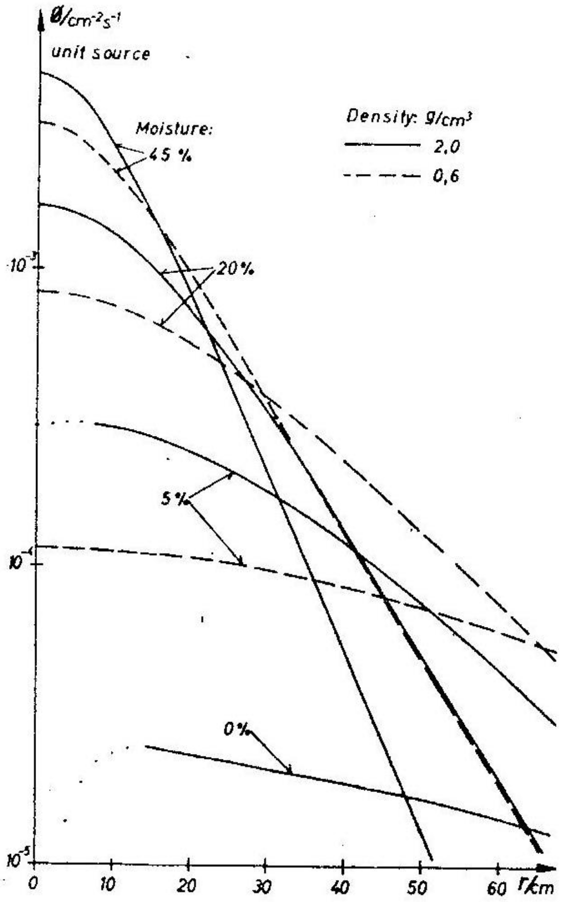

Fig. 3. Thermal flux.

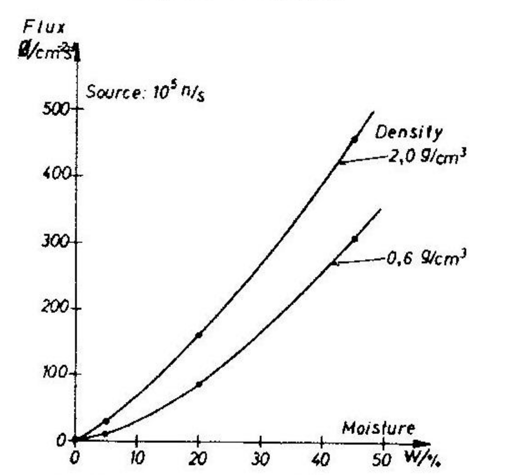

Fig. 4. Calibration curves at different densities.

are to be measured with samples of known composition and moisture. The strong dependence on dry density of the calibration curves has been observed experimentally by Unger and Claus [8].

The investigations concerning the influence of boron present in the material can be performed with the previous Monte Carlo results, because small amounts of this absorber change only the thermal cross sections remarkably. A multigroup calculation for the dry density of 2.0 g/cm^3 was carried out, and the influence of the additional absorber on the epithermal flux was found small as compared with the change in the thermal flux (fig. 5). For a moisture of 20% the relative changes per boron content in the thermal and epithermal fluxes in the origo were found to be 17.5%/0.005% and 0.5%/0.005%, respectively. Thus the effects of such additional absorbers as boron, cadmium etc. can be almost completely eliminated by the use of epithermal detectors like cadmium-covered BF_3-counters or cadmium-covered indium foils.

The numerical method of calculation rapported is applicable as well to other geometries, for instance to the reflection and transmission measurements on finite or semi-infinite slabs. It can also be modified so that computations for nonuniform distributions of moisture become possible.

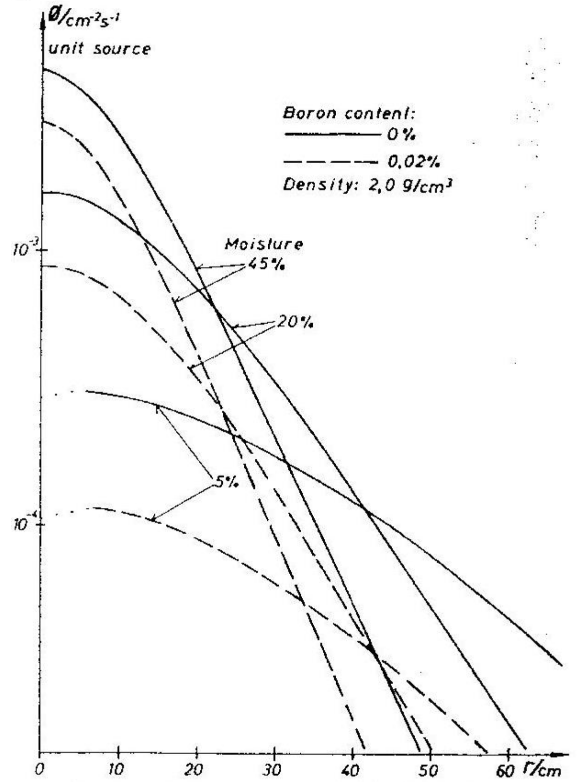

Fig. 5. Influence of boron on thermal fluxes.

3. MEASUREMENTS WITH AN IRON REFLECTOR AT THE BACK OF THE SOURCE

A series of measurements has been carried out to investigate the influence on the calibration curves of an additional iron reflector located behind the source. The arrangement is shown in fig. 6. The type of experiment was a surface measurement by Pawlin and Spinks [3] on a test material, which was fine sand of dry density 1.7 g/cm^3. The neutron source used was a Po-Be source of a strength of ~ 100 mC, and the detector was the BF_3-counter Philips ZP 1010 whose sensitivity is about 1 c/s per unit flux.

The resulting calibration curves obtained with and without the reflector are plotted in fig. 7. It is remarkable that the reflector does not raise the reading of the detector for dry material, but the sensitivity is considerably improved. This improvement is due to the reflection of inelas-

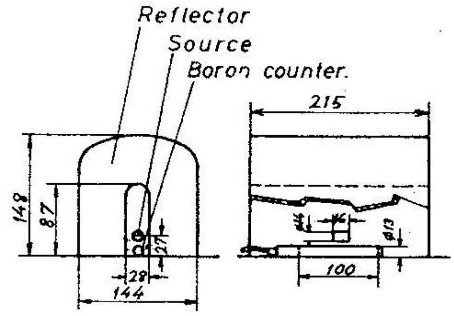

Fig. 6. Reflection measurement.

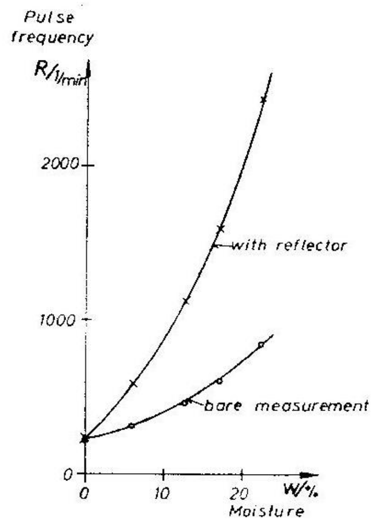

Fig. 7. Influence of reflector.

tically scattered neutrons from the iron, whereas the direction of motion of elastically scattered neutrons is mostly opposite. In the inelastic scattering the energy of the source neutrons decreases properly to the energies below 1 MeV so that they provide in the moisture measurement a better neutron source than the original source neutrons.

4. SOME MATHEMATICAL QUESTIONS OF THE ANALYSIS OF REFLECTION AND TRANSMISSION MEASUREMENTS ON SLABS

4.1. *The problem setting*

The aim of this section is to give a view on some basic ideas of the mathematical analysis of the reflection and transmission processes. The physical model subject to most of our considerations is diffusion theory for thermal neutrons and single-collision theory ("ray-theory") for fast neutrons. The ratio of the source term of thermal neutrons to the fast flux is assumed to be given by an effective cross section. This model is, of course, not totally justified in the analysis of the present problems, at least not when the moisture contert is low, but it is sufficient to give an insight into the main directions of a more complete analysis.

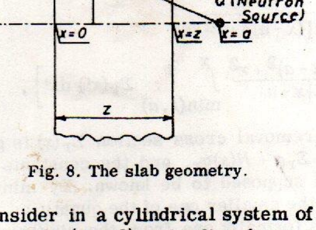

Fig. 8. The slab geometry.

We consider in a cylindrical system of coordinates a slab (fig. 8) extending from $x = 0$ to $x = z$, and to infinity in the radial direction. Suppose that a point source which emits isotropically Q_0 fast neutrons (energy E_0) per second is located in the point $x = a$, $r = 0$. We include all the possibilities $a > z$, $a = z$ or $a < z$. The slab is allowed to be stratified in the x direction in the sense that the content of hydrogen atoms per unit volume $N = N(x)$ is an arbitrary well-behaved function. Other atoms whose scattering and absorbing properties are accounted for are assumed to be space-independently distributed in the slab.

In general, the result of every measurement in the stationary case is a linear functional of the space, energy and direction dependent scalar flux $\varphi(\boldsymbol{r}, E, \boldsymbol{\Omega})$, which is the solution of the energy dependent Boltzmann equation [9] under appropriate boundary and source conditions. Specifically, if the effects subject to our observations are the reflection from the face $x = z$ and the transmission through the face $x = 0$ of neutrons with different energies and directions of motion, the functional is of the form

$$I = \int_{E_{th}}^{E_0} \mathrm{d}E \int_{\boldsymbol{n}\cdot\boldsymbol{\Omega}>0} \boldsymbol{n}\cdot\boldsymbol{\Omega}\, \mathrm{d}\Omega \int_{\substack{x=0\\x=z}} f(s, E, \boldsymbol{\Omega})\, \varphi(s, E, \boldsymbol{\Omega})\, \mathrm{d}s\,, \quad (1)$$

where the position vector $\boldsymbol{r}$ is denoted by s when lying on one of the surfaces and $\mathrm{d}s$ is the surface element. The vector $\boldsymbol{n}$ is the normal pointing outwards from the face under question. The function $f = f(s, E, \boldsymbol{\Omega})$ is an effectiveness function determined by the detector used. For instance, if we could select the neutrons from the infinitesimal energy interval $(E', E' + \mathrm{d}E)$ which are reflected from the surface element $\mathrm{d}s$ around s' into the infinitesimal solid angle $\mathrm{d}\Omega$ around the direction $\boldsymbol{\Omega}'$, the effectiveness would be the delta function $f = \delta(s, s')\delta(E - E')\delta(\boldsymbol{\Omega}, \boldsymbol{\Omega}')$. For other types of functionals associated with this problem of measurement, see ref. [10].

Following observations may be directly made concerning the evaluation of the functional in eq. (1), independently of the physical model used. If the solution of the linear Boltzmann equation or its approximation for the whole volume of the slab, $\varphi(\boldsymbol{r}, E, \boldsymbol{\Omega})$, is found, then (1) is easily obtained by substituting specifically $\boldsymbol{r} = s$. There exists, however, an alternative basing on the fact that the whole solution $\varphi(\boldsymbol{r}, E, \boldsymbol{\Omega})$ is actually not needed. The surface function is directly found in the method of invariant imbedding [9], in which the particular slab of thickness z is looked at as an element in the class of slabs of different thicknesses $z \geqslant 0$. In this technique, which is very useful in the description of stratified media, nonlinear initial value equations of Riccati type result for the reflection and transmission functions defined below in a special model. Furthermore, from the point of view of numerical computations the advantage of the reduction to initial value problems of boundary value problems wholly compensates the resulting nonlinearity.

The boundary function $\varphi(s,E,\Omega)$ naturally depends on the coefficients of the Boltzmann equation or, in the other formulation, of the equations of invariant imbedding. Because we keep the other functions as well as the boundary and source conditions fixed but think of different hydrogen densities $N(x)$ this function must be added to the factors determining φ, or $\varphi = \varphi(s,E,\Omega; N(x))$. Let T denote the generally nonlinear operator that carries $N(x)$ into the function $\varphi = \varphi(s,E,\Omega)$ defined in the proper phase space. If the functional of eq. (1) is written in the usual scalar product notation $I = [f,\varphi]$ and we wish to determine the change (variation) δI of I due to a little change $\delta N(x)$ of $N(x)$, we obtain for $\varphi = T(N(x))$:

$$\delta I = [f,\delta\varphi] = [f,\delta T(N(x))] = [f, T'(N(x))\delta N(x)]\,, \quad (2)$$

where $T'(N(x))$ is the Fréchet derivative

$$T'(N(x)) = \lim_{|\delta N(x)|\to 0} \frac{T(N(x)+\delta N(x)) - T(N(x))}{\delta N(x)}\,. \quad (3)$$

Note that the two first forms in eq. (2) are exact because of the linearity of I as a functional of φ, whereas for the last expression the requirement $|\delta N(x)| \ll |N(x)|$ must be fulfilled. Eq. (2) has very important applications, arising from the fact that in general the derivative $T'(N(x))$ is more easily formed than T itself. A usual principle of perturbation theory consists of the following. Suppose that the neutron balance equations cannot be analytically solved for φ with a particular $N(x)$. It is, however, often possible to find a comparison function $N_0(x)$ that yields an analytical solution and from which $N(x)$ deviates only slightly. (The simplest case is that $N(x)$ deviates little from a constant.) Then, by using eq. (2) the functional $I = I_0 + \delta I$ pertinent to $N(x)$ can be evaluated. Another application is that N may depend on a real parameter, time, but let the variation be so slow that the stationary neutron equations can be used. This is the case for drying of a concrete slab. For $N = N(x;t)$, $\mathrm{d}N = (\partial N(x;t)/\partial t)\mathrm{d}t$ we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \left[f, \frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] = \left[f, T'(N(x;t))\,\frac{\partial N(x;t)}{\partial t}\right]. \quad (4)$$

Thus, by making observations on $\mathrm{d}I/\mathrm{d}t$ an experimental relation between $T'(N(x))$ and $\partial N/\partial t$ is established. This may be used in either directions depending on whether scattering of neutrons or drying of concrete is concerned. It is most interesting to note how the partial derivative in eq. (4) can be further transformed by using the diffusion equation for the moisture [11] itself.

4.2. *Use of linear equations in the diffusion approximation*

It is assumed that in the description of the neutron balance for the slab of fig. 8 the complete Boltzmann equation mentioned in our introductory chapter may be replaced by the diffusion equation

$$L\Phi \equiv -\nabla\cdot D(x)\nabla\Phi(x,r) + \Sigma_a(x)\Phi(x,r) = q(x,r)\,, \quad (5)$$

where the thermal flux $\Phi = \Phi(x,r)$ is obtained from the scalar flux $\varphi(r,E,\Omega)$ by an integration over all directions of motion Ω and over the lowest, thermal part of the energy spectrum. Thus Φ depends only on the space coordinates x and r, if cylindrical symmetry is further assumed. The diffusion coefficient $D(x)$ is given by one third of the inverse transport cross section

$$D(x) = \frac{1}{3\Sigma_t(x)}\,,$$

where

$$\Sigma_t(x) = \Sigma_{t0} + N(x)\sigma_t\,,$$

and Σ_{t0}, σ_t are given constants. Likewise, $\Sigma_a(x) = \Sigma_{a0} + N(x)\sigma_a$, where Σ_{a0} and σ_a are given. The source term for thermal neutrons $q(x,r)$ is, as given by the ray theory of fast neutrons combined with the concept of removal cross section (see fig. 8):

$$q(x,r) = \frac{\Sigma_r(x)Q_0}{4\pi[(x-a)^2+r^2]} \exp\left[-\frac{(x-a)^2+r^2}{|x-a|}\int_{\min(z,a)}^{x} \Sigma_r(x')\,\mathrm{d}x'\right], \quad (6)$$

where the removal cross-section $\Sigma_r(x)$ is given by $\Sigma_r(x) = \Sigma_{r0} + N(x)\sigma_r$, and the constants Σ_{r0} and σ_r are supposed to be known. By $\min(z,a)$ we denote the smaller one of the quantities. The result (6) is formally one from the single-collision transport theory, but with respect to the production of thermal neutrons it is a highly artificial model that must be fitted to the experiments. It should be noted, however, as was mentioned in section 1, that in fast transmission measurements the transmitted fast intensity at distance r from the symmetry axis is in this approximation given by a geometrical factor times $\exp(-\int_z^0 N(x)\,\mathrm{d}x)$ which yields the integrated moisture directly. If the source is moved along the axis of symmetry within the slab (lower limit of the integral variable), there are possibilities for detailed determination of $N(x)$. In what follows only the thermal reflection experiment will be considered.

In the reflection experiment we assume that the functional of eq. (1) which gives the result of a single measurement depends only on the thermal flux (use of diffusion theory already implies independence on the direction of motion of neutrons) and is given by

$$I = \int_0^\infty f(r)\, J_{out}(z,r)\, 2\pi r\, dr$$

$$= \int_0^\infty f(r)\, [\tfrac{1}{4}\Phi(z,r) - \tfrac{1}{2}D(z)\Phi_x(z,r)]\, 2\pi r\, dr , \quad (7)$$

where $f(r)$ is the effectiveness "function" (also distributions allowed). The expression for the outgoing current J_{out} is one from the diffusion approximation. The functional of eq. (7) can be expressed by the use of the Parseval theorem for Hankel transforms also as follows:

$$I = 2\pi \int_0^\infty h\tilde{f}(h)\, \tilde{J}_{out}(z,h)\, dh , \quad (8)$$

where $\tilde{f}(h)$ and $\tilde{J}_{out}(z,h)$ denote the Hankel transforms

$$\tilde{f}(h) = \int_0^\infty J_0(hr)\, f(r)\, r\, dr ,$$

$$\tilde{J}_{out}(z,h) = \int_0^\infty J_0(hr)\, J_{out}(z,r)\, r\, dr \quad (9)$$

($J_0(x)$ is the Bessel function of first kind and zero order). Eq. (8) gives a linear functional in the space of transformed functions. Two special cases resulting from the choice $f_1(r) = 1$ or $f_2(r) = \delta(r - r_0)/r_0$, respectively, yield $I_1 = 2\pi\tilde{J}_{out}(z,0)$ and $I_2 = 2\pi \int_0^\infty J_0(r_0 h)\tilde{J}_{out}(z,h)h dh$. To evaluate of eq. (8) we must solve eq. (5) under the boundary conditions of diffusion theory

$$J_{in}(0,r) = \tfrac{1}{4}\Phi(0,r) - \tfrac{1}{2}D(0)\Phi_x(0,r) = 0 ,$$
$$J_{in}(z,r) = \tfrac{1}{4}\Phi(z,r) + \tfrac{1}{2}D(z)\Phi_x(z,r) = 0 , \quad (10)$$

which state zero ingoing thermal current from both faces. The next step consists of the reduction of eq. (5) with the boundary conditions (10) to a one-dimensional problem via the Hankel transform. The result is

$$-\frac{d}{dx} D(x) \frac{d\tilde{\Phi}(x,h)}{dx} + (\Sigma_a(x) + D(x)h^2)\tilde{\Phi}(x,h) = \tilde{q}(x,h) \quad (11)$$

with the associated transformed forms of eqs. (10). The Hankel transforms included in the previous equation have been defined analogously to eq. (9). We do not perform the transformation of q but assume that it is possible at least approximately. The solution of eq. (11) that satisfies the boundary conditions can be expressed in a standard manner

$$\tilde{\Phi}(x,h) = \int_0^z G_h(x,x')\, \tilde{q}(x',h)\, dx' \quad (12)$$

by using the Green's function $G_h(x,x')$ that satisfies

$$-\frac{d}{dx} D(x) \frac{dG_h(x,x')}{dx} + (\Sigma_a(x) + D(x)h^2)\, G_h(x,x') = \delta(x - x') , \quad (13a)$$

$$\tfrac{1}{4}G_h(0,x') - \tfrac{1}{2}D(0)\frac{dG_h}{dx}(0,x') = 0 \quad \text{for all } x' , \quad (13b)$$

$$\tfrac{1}{4}G_h(z,x') + \tfrac{1}{2}D(z)\frac{dG_h}{dx}(z,x') = 0 \quad \text{for all } x' . \quad (13c)$$

However, for our purposes we do not need $G_h(x,x')$ for all values of x. The reason for this is that the transformed outgoing (reflected) current $\tilde{J}_{out}(z,h)$ in the integral (8) is obtained as

$$\tilde{J}_{out}(z,h) = \tfrac{1}{4}\tilde{\Phi}(z,h) - \tfrac{1}{2}D(z)\tilde{\Phi}_x(z,h)$$

$$= \tfrac{1}{2}\int_0^z G_h(z,x')\, \tilde{q}(x',h)\, dx' \quad (14)$$

on the strength of eq. (12) and the transformed eqs. (10). Consequently the Green's function is actually needed only for $x = z$. Because the differential operator acting on $\tilde{\Phi}$ in eq. (11) is self-adjoint in the manifold defined by the transformed eqs. (10) the Green's function is symmetric in its arguments i.e. $G_h(z,x') = G_h(x',z)$. We must accordingly solve eq. (13) in the special case $x' = z$ and in the solution substitute x' for x. This problem proves to be identical with the solution of the following homogeneous equation with boundary conditions inhomogeneous at one end of the interval:

$$-\frac{d}{dx} D(x) \frac{dG_h(x,z)}{dx} + (\Sigma_a(x) + h^2)G_h(x,z) = 0 ,$$

$$\tfrac{1}{4}G_h(0,z) - \tfrac{1}{2}D(0)\frac{dG_h}{dx}(0,z) = 0 , \quad (15)$$

$$\tfrac{1}{4}G_h(z,z) + \tfrac{1}{2}D(z)\frac{dG_h}{dx}(z-\epsilon,z) = 1 .$$

Actually the second parameter z can be dropped out of the previous equations. For instance if the diffusion constant and the absorption cross section are space-independent the solution is easily obtained. From the definitions of the cross sections introduced in eq. (5) we get:

$$\delta\Sigma_a(x) = \sigma_a \delta N(x) , \quad \delta D(x) = -\frac{1}{3}\frac{\sigma_t}{\Sigma_t^2}\delta N(x) ,$$

$$\delta\Sigma_r(x) = \sigma_r \delta N(x)$$

as the small changes introduced in these functions by the little change $\delta N(x)$. If the operator in the first of eqs. (15) is denoted by L_{oh} and its change due to $\delta N(x)$ by δL_h

$$L_{oh} = -\frac{d}{dx} D_o(x) \frac{d}{dx} + \Sigma_{ao}(x) + D_o(x)h^2 , \tag{16}$$

$$\delta L_h = -\frac{d}{dx} \delta D(x) \frac{d}{dx} + \delta\Sigma_a(x) + \delta D(x)h^2 ,$$

but we restrict ourselves to variations $\delta D(0) = \delta D(z) = 0$ in order to remain in the same manifold of solutions, it follows from the condition $(L_{oh} + \delta L_h)(G_h + \delta G_h) = 0$ that $\delta G_h = -(G_h, \delta L_h G_h)$ when using the ordinary scalar product notation. If the variation $\delta\tilde{q}(x,h)$ is directly evaluated, we finally obtain the variation δI of eq. (8) as follows

$$\delta I \approx 2\pi \int_0^\infty h\tilde{f}(h) \tfrac{1}{2} \int_0^z [\delta G_h(x',z)\tilde{q}(x',h) + G_h(x',z)\delta\tilde{q}(x',h)] \, dx' \, dh , \tag{17}$$

where the expressions given above are to be substituted. This is a special case of eq. (2).

4.3. *Formulation of the problem via invariant imbedding*

In order to present new equations for the calculation of the transformed Green's function $G_h(x;z)$ of eq. (15) (the parameter z is here of special importance) we introduce the notations

$$u_h(x;z) = \tfrac{1}{4}G_h(x;z) - \tfrac{1}{2}D(x)\frac{dG_h(x;z)}{dx}$$

(right-going transformed current)

$$v_h(x;z) = \tfrac{1}{4}G_h(x;z) + \tfrac{1}{2}D(x)\frac{dG_h(x;z)}{dx}$$

(left-going transformed current)

so that the boundary conditions become $u_h(0;z) = 0$, $v_h(z;z) = 1$. The reduction of the first of eqs. (15) to a first-order system

$$\frac{du_h(x;z)}{dx} = -\left(\frac{1}{4D} + \tfrac{1}{2}\Sigma_a + \tfrac{1}{2}D(x)h^2\right) u_h(x;z) + \left(\frac{1}{4D} - \tfrac{1}{2}\Sigma_a - \tfrac{1}{2}D(x)h^2\right) v_h(x;z) , \tag{18a}$$

$$-\frac{dv_h(x;z)}{dx} = \left(\frac{1}{4D} - \tfrac{1}{2}\Sigma_a - \tfrac{1}{2}D(x)h^2\right) u_h(x;z) - \left(\frac{1}{4D} + \tfrac{1}{2}\Sigma_a + \tfrac{1}{2}D(x)h^2\right) v_h(x;z) \tag{18b}$$

subjected to the above conditions is quite classical and provides no difficulties. Of importance in the following are the coefficient functions which are here denoted by

$$a_h(x) = -\left(\frac{1}{4D(x)} + \tfrac{1}{2}\Sigma_a(x) + \tfrac{1}{2}D(x)h^2\right) ,$$
$$b_h(x) = \left(\frac{1}{4D(x)} - \tfrac{1}{2}\Sigma_a(x) - \tfrac{1}{2}D(x)h^2\right) . \tag{19}$$

They enter into the following Riccati equations for the reflection function $r_h(z) = u_h(z;z)/v_h(z;z) = u_h(z;z)$ and the transmission function $t_h(z) = v_h(0;z)/v_h(z;z) = v_h(0;z)$ as functions of the slab thickness z:

$$r_h'(z) = b_h(z) + 2a_h(z)r_h(z) + b_h(z)r_h^2(z) ,$$
$$r_h(0) = 0 ,$$
$$t_h'(z) = a_h(z)t_h(z) + b_h(z)r_h(z)t_h(z) , \tag{20}$$
$$t_h(0) = 1 .$$

As concerns the derivation of these formulae we must refer to the literature [9]. Use is made of the so-called Hadamard variational formula which gives the dependence of Green's functions on variations of the boundary (here thickness z).

For the case of a moisture distribution symmetrical with respect to $x = \tfrac{1}{2}z$ (whence reflection functions are independent of direction) it can be shown [9] that our Green's function is expressed in terms of the solutions of eqs. (20) as

$$G_h(x,z) = \frac{t_h(x)[1 + r_h(z-x)]}{1 - r_h(x)r_h(z-x)} . \tag{21}$$

In the case of a semi-infinite slab all formulae are considerably simplified. With the aid of the eqs. (20) the following formulae may be obtained for the variations $\delta r_h(z)$ and $\delta t_h(z)$ of the reflection and transmission functions, caused by the variations $\delta a_h(z)$ and $\delta b_h(z)$ which again are due to the variation $\delta N(z)$:

$$\delta r_h'(z) = \delta b_h(z) + 2\delta a_h(z)r_h(z) + \delta b_h(z)r_h^2(z) + [2a_h(z) + 2b_h(z)r_h(z)]\,\delta r_h(z) .$$

$$\delta t_h'(z) = \delta a_h(z)t_h(z) + \delta b_h(z)r_h(z)t_h(z) + [a_h(z) + b_h(z)r_h(z)]\,\delta t_h(z) + b_h(z)t_h(z)\delta r_h(z) , \tag{22}$$

when small terms of the second order have been dropped. These are linear equations for the variations sought and are more easily solvable than the complete eq. (20). Hence we can connect the variation of the hydrogen density $N(x)$ to the variations of the reflection and transmission functions, further to the variation of the Green's function (21), and finally to the variation δI of the functional (8). The generalization of the method of invariant imbedding to various formulations of the neutron transport theory is straight-forward.

4.4. *Remarks and conclusions*

One reason for the application of the physically unsatisfactory diffusion theory in this context was that the underlying methods, specifically that of invariant imbedding, prove certainly useful in the mathematical description of the diffusion of the moisture itself, in heat conduction etc. Many problems of space, time or concentration dependent diffusion coefficients etc. included in the reference [11] by Crank or [12, appendix 7] by Pihlajavaara may find their solutions by this method.

Very illustrative curves and tables which for various cases give the distribution of energy and direction of motion for neutrons reflected from a semi-infinite water medium, are found in our last reference [13].

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