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Error of Number of Radioactive Disintegrations

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Abstract: Statistics of exponentially decaying radiation of any nuclide always obeys the binomial stochastic distribution. The variance is derived for the number of atoms disintegrating in a time interval. The variance determines the error of the disintegrations number. When the measuring time is short compared with the half life, then the Poisson stochastics is valid. Today, computers can be used to determine either distribution very easily. The dependence of absolute and relative error on the measuring time interval is examined, but here assuming the absence of the background contribution.

Key words: Radioactivity, disintegration statistics, error.

1. Introduction

We see that the probability of radioactive disintegration is always described by the binomial distribution law [1-3]. The law is derived also here, as well as the absolute and relative errors of the count number it causes. The law is significant for relatively fast disintegrations. Then the error calculated using Poisson distribution can be considerably too large. Today the computers calculate also the binomial distributions rapidly.

We are assuming that N is the number atoms of a certain radioactive nuclide at time t = 0. On the disintegration of these nuclides there is the experimental result that the relative diminishing of the number is determined by the law:

$$\frac{dN}{N}\Big/dt = -\lambda \tag{1}$$

where, λ is the disintegration constant. Then, during a time interval T the number of the disintegrated nuclides is

$$n = N(1 - e^{-\lambda T}) \tag{2}$$

The number n, or some quantity proportional to it, can be measured. We have [4] $\lim_{T \to \infty} n = N$, Eq. (2).

2. Some Basic Theorems of Stochastic Functions

All the theorems we need for the stochastic variables u, v, .. have been presened by Blanc-Lapierre and Fortet [5]:

Theorem 1. The expectation of the sum of stochastic variables u, v, ..:$E(u + v + ..) = E(u) + E(v) + ..$, is the sum of the expectations of the variables.

Theorem 2. The variance of the sum of the independent random variables is the sum of the variances of the variables: $Var(u + v + \ldots) = Var(u) + Var(v) + \ldots$

Later we take the third theorem from Ref. [5].

3. Statistics of *n*

We thus have the group of N identical nuclides, at the time moment zero. We assume that the disintegrations of nuclides are independent from each others.

For a single nuclide we set the stochastic variable ν to present its disintegration during the time interval T. ν = 0, when the nuclide is not disintegrated, and ν = 1, when the disintegration occurs.

We set the probability p for to disintegrate (i.e. it "fails") and the probability q for not to disintegrate.

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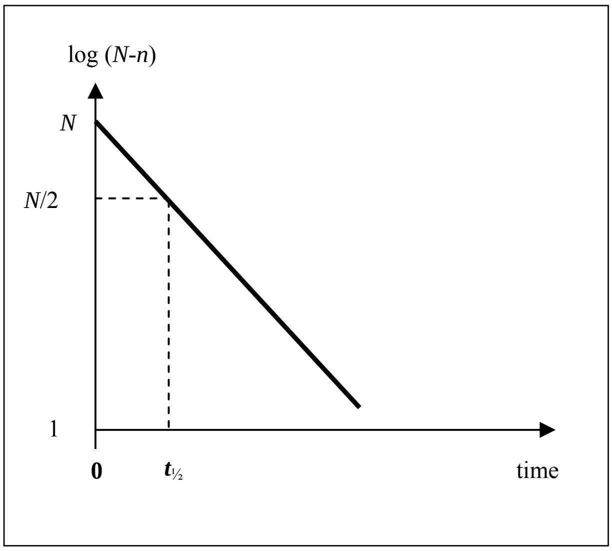

Fig. 1 Decay of N radioactive nuclides. n is the number of disintegration events. $t_{½}$ = log (2) λ^{-1}.

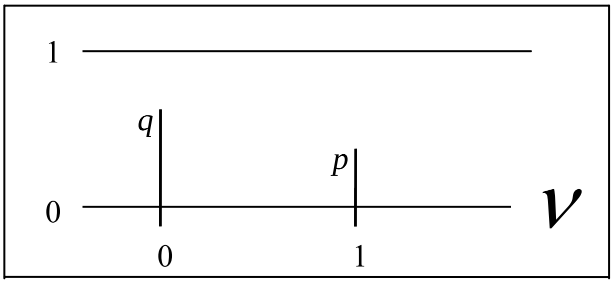

Fig. 2 ν is a two-valued (0 or 1) discrete variable. q and p are the probabilities of these values.

Rainwater and Wu [1] have the same structure for this Bernoulli stochastics. $p + q = 1$.

We take another integer stochastic variable r to present the number from the N nuclides to disintegrate during time T.

$$r = \sum_{i=1}^{N} \nu_i \,,\ \ r \in \{0, \ldots, N\} \qquad (3)$$

How many ways a certain number r of atoms comes to be disintegrated? N atoms can be the first to disintegrate. $N - 1$ is the number of atoms left, and each of them can be the atom to disintegrate as the second, $N - 2$ as the third, etc. $N - r + 1$ is the number of atoms that can disintegrate lastly. $V = N(N - 1)(N - 2)\cdots(N - r + 1)$, variation, is the number for all possible ways of the r atom disintegrations from N atoms. The order of atom disintegrations is totally random. Take a group of the r disintegrated atoms. Each of these atoms can be in any position in V. Therefore, the r atoms can be ordered randomly in the number $r!$, permutation, ways. And the number V should be divided by $r!$ and the result is the number

$$\binom{N}{r} = \frac{V}{r!} = \frac{N!}{(N-r)!\,r!}\,, \qquad (4)$$

combination, the number of the ways, in which the r disintegrations can be occurred among the N atoms [6].

Because $\nu_i \in \{0, 1\}$ with the probabilities q or p, we have the probability

$$P(r) = \binom{N}{r} p^r q^{N-r} = \binom{N}{r} p^r (1-p)^{N-r} \qquad (5)$$

of the r disintegrations. $r = \nu$ when $N = 1$. We see that the probability of radioactive disintegration is described by the binomial distribution law [1-3].

For a radioactive nuclide we postulate the probabilities $p = 1 - e^{-\lambda T}$ and $q = e^{-\lambda T}$, for the disintegration during the time T. In some references you can find the derivation of $q = \exp(-\lambda T)$, but it implies the assumptions of the Poisson stochastics (demands $\lambda T \ll 1$).

The mean value of disintegration of one nuclide (during T) is the expectation:

$$E(\nu) = \overline{\nu} = q0 + p1 = p = 1 - e^{-\lambda T} \qquad (6)$$

and the variance of its disintegration is

$$Var(\nu) = D^2(\nu) = \sigma^2(\nu) = \mathrm{E}(\nu - \overline{\nu}\,)$$
$$= (0 - p)^2 q + (1 - p)^2 p \qquad (7)$$

This holds for every single nuclide of the N atoms. Because (Theorem 1) the expectation of the sum of stochastic variables $u, \nu, ..$: $E(u + \nu + ..) = E(u) + E(\nu) + ..$, is the sum of the expectations of the variables, for the stochastic variable r we have:

$$E(r) = E(\sum_{i=1}^{N} \nu_i) = \sum_{i=1}^{N} E(\nu_i) = Np = N(1 - e^{-\lambda T}) = n. \qquad (8)$$

This expectation agrees with Eq. (2).

The expectation can be shown also explicitly:

$$E(r) = \sum_{r=0}^{N} rP(r) = \sum_{r=1}^{N} r\binom{N}{r} p^r (1-p)^{N-r}$$

$$= Np\sum_{s=0}^{N-1}\binom{N-1}{s}p^s(1-p)^{N-1-s}$$

$$= Np = n\,,\ (s = r - 1) \tag{9}$$

because $\sum_{k=0}^{N}\binom{N}{k}p^k(1-p)^{N-k} = 1$ for any N.

Because the disintegration events are independent, the variance of r (Theorem 2) is

$$\begin{aligned} Var(r) &= \sigma^2(r) = \sigma^2(N\nu) = N\sigma^2(\nu) \\ &= N\,pq = n\,e^{-\lambda T} \end{aligned} \tag{10}$$

Rainwater and Wu[1] present the explicit derivation. The mean statistical error [1-3]

$$\Delta n = \sigma(r) = \sqrt{n}\,e^{-\lambda T/2} \tag{11}$$

You see, this value is smaller than the error value $\sqrt{n}$ in the case $\lambda T \cong 0$.

Also the relative error

$$\frac{\Delta n}{n} = \frac{e^{-\lambda T/2}}{\sqrt{n}} \tag{12}$$

has the same exponential function multiplier, that is smaller than 1. The relative error Eq. (12) is in practice very important, but in the measurements the background is to be concerned.

4. Binomial and Poisson Statistics

When $\lambda T \cong 0$, then $p = 1 - e^{-\lambda T}$ is very small, i.e. λT . The limit of $P(r)$ has been sought under this assumption and holding

$$pN = n \cong \lambda TN \tag{13}$$

is constant. When p decreases, N must increase. The limit:

$$\lim_{\substack{p\to 0\\ pN=n}} P(r) = \lim_{\substack{N\to\infty\\ pN=n}} \frac{N(N-1)\cdots(N-r+1)}{r!}\,p^r(1-p)^{N-r}$$

$$= \lim_{\substack{N\to\infty\\ pN=n}} \frac{N^r(1-\frac{1}{N})\cdots(1-\frac{r-1}{N})}{r!}\,p^r(1-\frac{n}{N})^{N(1-\frac{r}{N})}$$

$$= \lim_{\substack{N\to\infty\\ pN=n}} \frac{(pN)^r}{r!}(1-\frac{n}{N})^N = \frac{n^r}{r!}e^{-n} \tag{14}$$

is the Poisson distribution. The limit is approached uniformly for all r as presented by Feller [7]. Also now $E(r) = n$, but $Var(r) = n$. The Poisson distribution has been much used for approximations of $P(r)$.

Today, the programs of computing give as easily the stochastic distributions, binomial and Poisson (Fig. 3), but the binomial one is always valid. Gol'danskii et al. [2], illustrate the binomial distributions $n = 5$ for $N = 6$, 10, 20, and ∞, the Poisson distribution.

5. Background Often from Poisson Distributed Nuclides

We can suppose that the nuclide distributions are independent. The background n_b can often only consist of radiation of the nuclides for which the Poisson statics is valid. From Ref. [5] we find:

Theorem 3. The stochastic function of the sum of the independent random variables, which have the Poisson stochastic functions, is a Poisson stochastic function.

1,2,…,k are such nuclides and n_i presents the count number of the nuclide i. Then the background $n_b = \sum_{1}^{k} n_i$ is Poisson distributed, too.

6. Error Considerations

Suppose a 4π detector, which can detect all disintegrations that occur for the N nuclides. The number

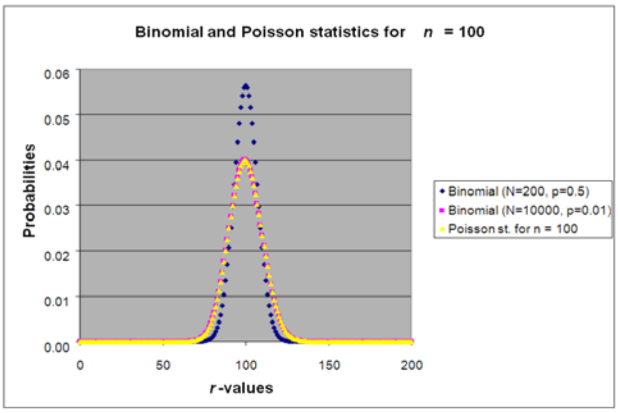

Fig. 3 Binomial and Poisson statistics. Distributions for *n* = 100.

N can be known or estimated. We shall determine T in 2 cases of demands for error, starting from Eq. (11).

1. $\Delta n = \varepsilon$

Then:

$$\varepsilon = \sqrt{n}\, e^{-\lambda T/2} = (N(1-e^{-\lambda T}))^{\frac{1}{2}} e^{-\lambda T/2} \quad (15)$$

Of the two solutions for T from Eq. (15) we consider

$$T = t_{½} \left[\frac{\log\left(\frac{N}{2\varepsilon^2}(1+\sqrt{1-4\varepsilon^2/N})\right)}{\log 2} \right] \quad (16)$$

For the error $\varepsilon = \pm 1$, or the case when the last disintegration starts to be probable, we find values in Table 1 (The other solution of Eq. (16), when $(1-\sqrt{1-4\varepsilon^2/N})$ in the argument of the natural logarithmic function, is in the third column).

T of Eq. (2), for $n = N - 1$, and from Eq. (16) for large N have the same values as in Table 1, but for small N, they differ, e.g., for $N = 100$ the $T/t_{½}= 6.629$ from Eq. (16), but from Eq. (2) $n = N - 1 = 99$ when $T/t_{½} = \log(N)/\log(2) = 6.644$.

2. $\frac{\Delta n}{n} = \beta$

For practice this error is very useful. It is:

$$\beta = e^{-\lambda T/2} / \sqrt{n} = (N)^{-\frac{1}{2}} (e^{\lambda T} - 1)^{-\frac{1}{2}} \quad (17)$$

This errordecreases monotonously for all the time andis zero when time isinfinite. Further:

$$\lambda T = \log(1 + \frac{1}{\beta^2 N}) \quad (18)$$

The larger N, the smaller Eq. (18) and T are. If $N = 10^4$ and $\beta = 1\%$, then $\lambda T = \log 2$, or $T = t_{½}$.

The number of counts, in practical measurements, is proportional to n, but is often much smaller. We have not considered the effects of the background, and the dead time that influence in big counting rates. The significant background causes, that the optimal measuring time is also determined by the portion of background and how background alters with time.

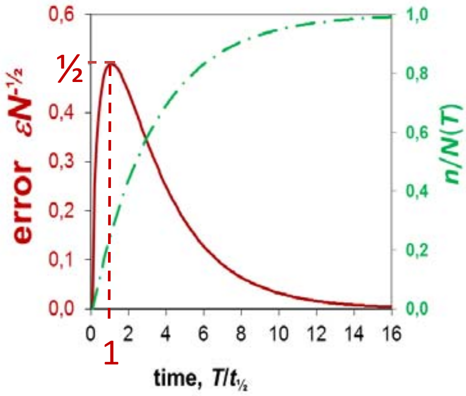

Fig. 4 Presentation of the error ε of Eq. (15), and also the curve of $n(T)$ from Eq. (2).

Table 1 Measuring times T for error $\varepsilon = \pm 1$.

N	$T/t_{1/2}$	
10^4	13.3	$1.44\ 10^{-4}$
10^6	19.9	$1.44\ 10^{-6}$

7. Bayesian Approach

Often the number N is not known. The result of measurement of $n(T)$ is a certain value of r. The Bayesian approach of our stochastic problem is: what a single result $n$tells of its mean value? For the case $T \ll t_{½}$ Rainwater and Wu [1] wrote Eq. (14) (n instead of r) as a stochastic function:

$$P(u) = \; = \frac{u^n}{n!}\exp(-u) \quad (19)$$

for the mean value u. $E(u) = n + 1$ is then the best estimate for the mean value of n [1-3, 8].

Stevenson [8] has derived the solutions for the general binomial distribution, where $T \approx T_{½}$. Though he assumed for $N \geq n$ roughly that each N has the same probability, he found the good results:

$$E(r) = n + 1 - p$$

where in the Poisson case $p \cong 0$. When $p = 1$, the result agrees with Eq. (8). When approaching this limit then in the Fig. 1, T is much larger than $t_{½}$.

$$Var(r) = (1 - p)(n + 1) \quad (20)$$

When $p \to 1$, i.e., $T \to \infty$, then $Var(r) \to 0$ means that n approaches the exact value N, i.e. without the error. However, Eq. (11) is for $T \gg t_{1/2}$ the more relevant presentation of error than the square root of Eq. (20).

8. Discussion

The probability p is above "postulated" as a very generally assumed presentation. Can it have another presentation? Rainwater and Wu [1] needed no presentation for p and, however, obtained the very general results.

Often the number of radioactive nuclides N is unknown. Today in certain cases it may be measured. I think the mass pectrometers should be very applicable for this. Then for certain N and T we can determine the stochastic distribution of $n(T)$ experimentally.

9. Conclusions

We have considered the exponential attenuation of radioactive substance. We have assumed that all the disintegrations of its nuclides are measured. The binominal distributions of the number of the disintegrations are illustrated in Fig. 3 (for the mean value n = 100). You see that the distribution is more narrow when the disintegration probability p is larger (When p = 0.5, then the measuring time $T = t_{½}$). When p (and T) increases then the variance of the distribution and the error of n approach the value zero.

The error of the number in counts is the largest when the time of measurement T is equal to the half life $t_{½}$ (Fig. 4). For all the time the relative error$\Delta n/n$ is diminishing with increasing time.The error obeys the Poisson statistics only when $T << t_{1/2}$. The binomial statistics the error obeys always. Today the computers calculate the binomialerror easily.

We have shown that $E(r) \in [n, n+1]$. When N is unknown and $T << t_{1/2}$, then the mean $E(r) \cong n + 1$ and the error is$\sqrt{n+1}$. Otherwise the mean $E(r) = n$and the error is $\sqrt{n}\exp(-\lambda T/2)$, Eq. (11) (Eq. (20) can be incorrect). When $T >> t_{1/2}$, then the error gradually approaches the value zero.

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